# On the Invariance of Distribution Functions

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The Lorentz invariance of the fine-grained one-particle distribution function of a gas, as well as that of multiple-time many-particle distribution functions, is shown to be a consequence of the very definition of a physical event, so that it requires no formal proof. Since it is independent of any special property of Lorentz transformations, this invariance holds quite generally for arbitrary transformations of space-time coordinates.

There have been many unsuccessful attempts to prove the Lorentz invariance of the one-particle distribution function of a gas of interacting particles. (For a detailed list of such attempts and their refutations, see Ref. [1].) Eventually, it was shown [1, 2] that the fine-grained one-particle distribution function, defined by

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = \sum_n \delta(\boldsymbol{r} - \boldsymbol{r}_n(t))\, \delta(\boldsymbol{p} - \boldsymbol{p}_n(t)) \qquad (1)$$

or by an ensemble average of such expressions, is invariant under Lorentz transformations. Here $\boldsymbol{r}$, $\boldsymbol{p}$, $t$ denote position, momentum, and time, $\boldsymbol{r}_n(t)$ and $\boldsymbol{p}_n(t)$ are the position and momentum vectors of particle $n$ at time $t$, and the sum is over all particles of the gas. In order to prove this invariance, van Kampen [1] carried out explicitly the corresponding Lorentz transformations, whereas Piña and Balescu [2] based their proof on the generators of the Lorentz group. It is the aim of the present paper to show that the invariance of function (1) is a simple fact that can be proved by physical arguments without having recourse to any mathematical formalism.

Obviously, the function $f$, Eq. (1), provides a complete, microscopic description of the gas, or, to put it differently, it describes for an observer $O$ all physical events at all space-time points of the gas. For example, the function $f$ may describe, among others, the events that at space-time point $(\boldsymbol{r}_a, t_a)$ there is no particle present, at $(\boldsymbol{r}_b, t_b)$ one particle with momentum $\boldsymbol{p}_0$, and at $(\boldsymbol{r}_c, t_c)$ two (just colliding) particles with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Now consider another observer $O'$ in relative motion with respect to $O$, whose distribution function is defined by

$$f'(\boldsymbol{r}', \boldsymbol{p}', t') = \sum_n \delta(\boldsymbol{r}' - \boldsymbol{r}_n'(t'))\, \delta(\boldsymbol{p}' - \boldsymbol{p}_n'(t'))\,. \qquad (2)$$

For $O'$, the same events are described by saying that at $(\boldsymbol{r}_a', t_a')$ there is no particle, at $(\boldsymbol{r}_b', t_b')$ one particle with momentum $\boldsymbol{p}_0'$, and at $(\boldsymbol{r}_c', t_c')$ two particles with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$, and these events are contained in $f'$, as $f$ and $f'$ are identical in form. Quite generally, all events contained in $f$ are likewise contained in $f'$, merely expressed in another language, and vice versa. It follows that the distribution function $f$ is invariant,

$$f'(\boldsymbol{r}', \boldsymbol{p}', t') = f(\boldsymbol{r}, \boldsymbol{p}, t) = \text{inv.} \qquad (3)$$

To be specific: Observer $O$ $[O']$ finds, for example, that for $(\boldsymbol{r}_b, t_b)$ $[(\boldsymbol{r}_b', t_b')]$ the function $f$ $[f']$ vanishes for all $\boldsymbol{p}$ $[\boldsymbol{p}']$ except for $\boldsymbol{p} = \boldsymbol{p}_0$ $[\boldsymbol{p}' = \boldsymbol{p}_0']$ where precisely one delta function is different from zero. In order to obtain from the (infinite) delta function the (finite) number of particles, observer $O$ $[O']$ has to integrate $f$ $[f']$ over a small, six-dimensional volume $d^3r\, d^3p$ $[d^3r'\, d^3p']$ with $t = t_b = \text{const.}$ $[t' = t_b' = \text{const.}]$ and centered at $(\boldsymbol{r}_b, \boldsymbol{p}_0)$ $[(\boldsymbol{r}_b', \boldsymbol{p}_0')]$. In spite of the fact that these integration volumes are different (since they lie in different hyperplanes, $t = \text{const.}$ and $t' = \text{const.}$, respectively), both observers find the same result (namely unity) because their respective delta functions are different from zero only at the *same* event-point of the seven-dimensional $(\boldsymbol{r}, \boldsymbol{p}, t)$-space which belongs to both $d^3r\, d^3p$ and $d^3r'\, d^3p'$.

Provided that a classical description by means of point-like particles is adequate, the invariance property (3) holds for particles and photons alike, and is independent of whether or not the number of particles is conserved. Clearly, the validity of Eq. (3) is not influenced by averaging over a statistical ensemble of independent systems.

As we have seen, the validity of Eq. (3) is independent of any special property of Lorentz transformations, which are needed only for calculating explicitly the primed quantities from the unprimed ones, and vice versa. It follows, then, that the invariance property (3) holds quite generally for

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arbitrary transformations of the space-time coordinates, as considered in general relativity.

Our reasoning also shows immediately that fine-grained multiple-time many-particle distribution functions like $f^{(2)}(\boldsymbol{r}_1, \boldsymbol{p}_1, t_1; \boldsymbol{r}_2, \boldsymbol{p}_2, t_2)$ are invariant, in constrast to single-time distribution functions like $\bar{f}^{(2)}(\boldsymbol{r}_1, \boldsymbol{p}_1; \boldsymbol{r}_2, \boldsymbol{p}_2; t)$ [1], because in the latter case different observers observe in general different events. If the particles have spin (electric moment, magnetic moment, etc.) $\boldsymbol{s}$, the $\mu$-space must be enlarged from $(\boldsymbol{r}, \boldsymbol{p})$ to $(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{s})$, and in the fine-grained distribution functions must be included factors $\delta(\boldsymbol{s} - \boldsymbol{s}_n(t))$ in order to make them invariant [3].

The invariance of the fine-grained one-particle distribution function $f$ can thus be traced back to the fact that here different observers deal with the same, strictly localized events. This should be contrasted with the coarse-grained one-particle distribution function $\bar{f}$, which is defined such that $\bar{f}(\boldsymbol{r}, \boldsymbol{p}, t)\, \Delta^3 r\, \Delta^3 p$ is the number of particles, at time $t$, in a small, but *finite* volume $\Delta^3 r\, \Delta^3 p$ of $\mu$-space centered at $(\boldsymbol{r}, \boldsymbol{p})$. In the general case of interacting particles, $\bar{f}$ is *not* invariant [1] because observers $O$ and $O'$ use *different* events for counting this number, owing to the fact that events that are simultaneous for $O$ are not simultaneous for $O'$, and vice versa.

[1] N. G. van Kampen, Physica **43**, 244 (1969).
[2] E. Piña and R. Balescu, Acta Phys. Austr. **28**, 309 (1968).
[3] Ch. G. van Weert, Proc. Kong. Ned. Akad. Wetensch. B **73**, 381 (1970).